J. DIFFERENTIAL GEOMETRY
8 (1973) 103-106

A CHARACTERIZATION OF RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE

RICHARD HOLZSAGER

As in [2], consider the parallel bodies of a hypersurface in a Riemannian manifold. That is, suppose M is a submanifold of codimension 1 with oriented normal bundle in a manifold $\bar{M}$. Define a homotopy $h: M \times R \to \bar{M}$, by letting $h(x, t) = \gamma_x(t)$, where γ_x is the geodesic through x whose tangent at x is the positive (with respect to the orientation on the normal bundle of M) unit normal vector. In other words, $h_t(M)$ is obtained by translating M distance t along orthogonal geodesics.

If M is a compact hypersurface (with or without boundary), it makes sense to consider the area (or volume) $A_M(t)$ of the singular hypersurface M_t. If M and $\bar{M}$ are C^∞, so is $A_M: R \to R$. In [1], we showed that surfaces of constant curvature c are characterized by the fact that for any hypersurface (i.e., curve) A_M satisfies the differential equation $A'' + cA = 0$. This result is now generalized to higher dimensions.

Theorem. *For an n-dimensional C^∞ Riemannian manifold $\bar{M}$, there is a differential equation $A^{(n)} + a_1A^{(n-1)} + \cdots + a_nA = 0$ (a_i constant) satisfied by A_M for every hypersurface M if and only if $\bar{M}$ has constant sectional curvature. The relation between the equation and the curvature is* $c = a_2 \Big/ \binom{n+1}{3}$.

Remark. It is impossible for an equation of order m less than n to be satisfied by every A_M. To show this choose some $x \in M$ and an orthogonal base $T_1, \cdots, T_n$ for the tangent space at x. Define a coordinate system ϕ_m about x by $\phi_m(r_1, \cdots, r_n) = \exp_y X$, where $y = \exp_x \sum_{m+2}^n r_iT_i$, and X is the parallel translation of $\sum_1^{m+1} r_iT_i$ to y along $\exp_x t \sum_{m+2}^n r_iT_i$. Let U be a small neighborhood of $(1, 0, \cdots, 0)$ in an m-sphere S^m, and V a small neighborhood of the origin in an $(n - m - 1)$-dimensional Euclidean space R^{n-m-1}. For small values of t, ϕ_m will imbed $(tU) \times V$ in $\bar{M}$, $\phi_m((tU) \times V)$ forms a family of "parallel" hypersurfaces and $A(t) = t^m \int \sqrt{g} \circ \phi \circ (t \times \mathrm{id})\, d\,\mathrm{Vol}$, integral over $U \times V$, where g is the determinant of the metric tensor on $\bar{M}$ with respect to ϕ. Then $A^{(m)}(0) = m!\,\mathrm{Vol}\,(U \times V)$, $A^{(i)}(0) = 0$ for $i < m$. Thus A cannot satisfy an equation of order m.

Received October 12, 1971.

Proof of the theorem. Assume the equation is satisfied by every A_M. Let $\phi = \phi_{n-2}$ be as in the remark (i.e., build a coordinate system using tubes about the geodesic through T_n). Let

$$A_0(t) = \lim A(t)/\mathrm{Vol}\,(U \times V) = t^{n-2}\sqrt{g}\,(\phi(1, 0, \cdots, 0)) ,$$

limit taken as U and V converge down to $(1, 0, \cdots, 0) \in S^{n-2}$ and $0 \in R$ respectively. A_0 will also satisfy the equation, giving for $t = 0$

$$(1) \qquad \binom{n}{2}(n-2)!\,T_1T_1\sqrt{g} + (n-1)(n-2)!\,a_1T_1\sqrt{g} + (n-2)!\,a_2\sqrt{g} = 0 ,$$

where we write T_i for $\partial/\partial x_i$ throughout the coordinate system.

Let us also write D_i for covariant differentiation with respect to T_i. Then, as in [2], $T_1\sqrt{g} = \sum_{i=1}^{n} \gamma_{ii}\sqrt{g}$, where $D_1T_i = \sum_j \gamma_{ij}T_j$ and $T_1T_1\sqrt{g} = (\sum_{i,j=2}^{n} \gamma_{ii}\gamma_{jj} + \sum_i T_1\gamma_{ii})\sqrt{g}$. By definition of ϕ, $\sum_{i=1}^{n-1} r_iD_i\,(\sum_{j=1}^{n-1} r_jT_j) = 0$ at any point of the form $(tr_1, \cdots, tr_{n-1}, r_n)$. In particular, this implies $D_iT_j = 0$ for all $i, j \leq n-1$ at x. Also, $D_nT_i = 0$ for all i, so $D_1T_n = D_nT_1 = 0$. Consequently $\gamma_{ii}(x) = 0$ for all i. Thus

$$(2) \qquad \binom{n}{2}\sum_i T_1\gamma_{ii} + a_2 = 0 \qquad \text{at } x .$$

$$\begin{aligned} T_1\gamma_{ii} &= \sum_j (T_1\gamma_{ij})\langle T_i, T_j\rangle = T_1\Big(\sum_j \gamma_{ij}\langle T_i, T_j\rangle\Big) - \sum_j \gamma_{ij}T_1\langle T_i, T_j\rangle \\ &= T_1\langle T_i, D_1T_i\rangle = \langle D_1T_i, D_1T_i\rangle + \langle T_i, D_1D_1T_i\rangle = \langle T_i, D_1D_iT_1\rangle , \end{aligned}$$

so (2) becomes

$$(3) \qquad \binom{n}{2}\sum_i \langle T_i, D_1D_iT_1\rangle + a_2 = 0 .$$

At $\phi(x_1, \cdots, x_n)$, $\sum_{j,k=1}^{n-1} x_jx_kD_jT_k = 0$. Applying D_i $(i = 1, \cdots, n-1)$ at $\phi(x_1, 0, \cdots, 0)$ gives $2\,x_1D_iT_1 + x_1^2D_iD_1T_1 = 0$. Dividing by x_1 and applying D_1 give $2D_1D_iT_1 + D_iD_1D_1 + x_1D_1D_iD_1T_1 = 0$, so $D_iD_1T_1 = -2D_1D_iT_1$ at x. Therefore the sectional curvature determined by T_1 and T_i at x for $i = 1, \cdots, n-1$ is $R(1, i) = -3\langle D_1D_iT_1, T_i\rangle$. Also, $R(1, n) = -\langle D_1D_nT_1, T_n\rangle$, since D_1T_1 vanishes along $\phi(0, \cdots, 0, t)$, so $D_nD_1T_1 = 0$ at x. Now (3) becomes

$$(4) \qquad R(1, n) + \frac{1}{3}\sum_{2}^{n-1} R(1, i) = a_2\Big/\binom{n}{2} .$$

The roles played in this whole argument by T_n and T_i $(i = 2, \cdots, n-1)$ may be switched, adding $\frac{2}{3}(R(1, i) - R(1, n))$ to the left side without changing the right. Thus $R(1, i) = R(1, n)$, so $\frac{n+1}{3}R(1, n) = a_2\Big/\binom{n}{2}$, or $R(1, n) = a_2\Big/\binom{n+1}{3}$. Since x, T_1, T_n were arbitrary, this finishes the proof in one direction.

Now assume $\overline{M}$ has constant curvature. For any tangent V to M at x, V has a canonical extension along the orthogonal geodesic ($V(h_t(x)) = dh_t(V)$), so if T is the unit normal vector, then $D_T V$ makes sense. Note that if W is another tangent to M at x, $\langle D_T V, W\rangle = \langle D_T W, V\rangle$. To see this, a coordinate system ϕ in $\overline{M}$ about x is said to be allowable if it is obtained by taking a coordinate system ψ in M about x and setting $\phi(r_1, \cdots, r_n) = h_{r_1}(\psi(r_2, \cdots, r_n))$. If T, V, W are extended to have constant components in an allowable coordinate system, then $[V, W] = [T, V] = [T, W] = \langle T, V\rangle = \langle T, W\rangle = 0$, so

$$\begin{aligned}\langle D_T V, W\rangle &= \langle D_V T, W\rangle = -\langle T, D_V W\rangle = -\langle T, D_W V\rangle \\ &= \langle D_W T, V\rangle = \langle D_T W, V\rangle .\end{aligned}$$

Further, applying T to the relation $\langle D_T V, W\rangle = \langle D_T W, V\rangle$ gives $\langle D_T D_T V, W\rangle = \langle D_T D_T W, V\rangle$.

Since $\langle D_T V, W\rangle$ is symmetric and bilinear in V and W, it is possible to choose an orthonormal base $T_2, \cdots, T_n$ for the tangent space to M at x such that $\langle D_T T_i, T_j\rangle = 0$ for $i \neq j$, and also an allowable coordinate system so that at x $\partial/\partial x_i = T_i$ for $i = 1, \cdots, n$ (where we now write T_1 for T). If V is a linear combination of $T_2, \cdots, T_n$ at any point of the coordinate neighborhood and R is the curvature tensor, then $\langle R(T_1, V)T_1, V\rangle = \langle D_V D_1 T_1, V\rangle - \langle D_1 D_V T_1, V\rangle = -\langle D_1 D_V T_1, V\rangle = -\langle D_1 D_1 V, V\rangle$ since $D_1 T_1$ is identically 0. If c is the sectional curvature, then since $\langle T_1, V\rangle = 0$ and $\langle T_1, T_1\rangle = 1$, $c = -\langle D_1 D_1 V, V\rangle / \langle V, V\rangle$. Thus, as quadratic forms on the span of $T_2, \cdots, T_n$ at any point, $\langle D_1 D_1 V, V\rangle$ is equal to $\langle -cV, V\rangle$. The symmetric bilinear forms $\langle D_1 D_1 V, W\rangle$ and $\langle -cV, W\rangle$ are equal, so $\langle D_1 D_1 T_i, T_j\rangle = \langle -cT_i, T_j\rangle$ for $i, j \geq 2$. Since also

$$\begin{aligned}\langle D_1 D_1 T_i, T_1\rangle &= T_1\langle D_1 T_i, T_1\rangle - \langle D_1 T_i, D_1 T_1\rangle = T_1\langle D_i T_1, T_1\rangle \\ &= \tfrac{1}{2} T_1 T_i \langle T_1, T_1\rangle = 0 = \langle -cT_i, T_1\rangle ,\end{aligned}$$

it follows that $D_1 D_1 T_i = -cT_i$ for $i = 2, \cdots, n$.

Next note that $c\langle T_i, T_j\rangle + \langle D_1 T_i, D_1 T_j\rangle$ is constant along $h_t(x)$ $(i, j \geq 2)$, since

$$\begin{aligned}&T_1(c\langle T_i, T_j\rangle + \langle D_1 T_i, D_1 T_j\rangle) \\ &\quad = c\langle D_1 T_i, T_j\rangle + c\langle T_i, D_1 T_j\rangle + \langle D_1 D_1 T_i, D_1 T_j\rangle + \langle D_1 T_i, D_1 D_1 T_j) = 0 .\end{aligned}$$

But at x, $\langle D_1 T_i, T_j\rangle = 0$ for $j \neq i$, so $D_n T_i$ is a multiple of T_i for $i = 2, \cdots, n$, so $c\langle T_i, T_j\rangle + \langle D_1 T_i, D_1 T_j\rangle = 0$ at x and consequently at $h_t(x)$. Thus

$$T_1T_1\langle T_i, T_j\rangle = \langle D_1D_1T_i, T_j\rangle + 2\langle D_1T_i, D_1T_j\rangle + \langle T_i, D_1D_1T_j\rangle$$
$$= -4c\langle T_i, T_j\rangle \;,$$

for $i, j \geq 2$, $i \neq j$. This second order equation, together with the initial conditions $\langle T_i, T_j\rangle = T_1\langle T_i, T_j\rangle = 0$ at x, implies that $\langle T_i, T_j\rangle$ is identically 0 along $h_t(x)$. Therefore $|dh_t(T_2 \wedge \cdots \wedge T_n)| = \prod_2^n |T_i|$ at $h_t(x)$. Now

$$T_1T_1|T_i| = -c|T_i| + (\langle T_i, T_i\rangle\langle D_1T_i, D_1T_i\rangle - \langle D_1T_i, T_i\rangle^2)/|T_i|^3$$
$$= -c|T_i|$$

(D_1T_i being a multiple of T_i). This means that $|T_i|$ is a linear combination of $\sin\sqrt{c}\,t$ and $\cos\sqrt{c}\,t$ or of $\sinh\sqrt{-c}\,t$ and $\cosh\sqrt{-c}\,t$ or of 1 and x, depending on whether c is positive, negative or 0. Therefore $|dh_t(T_2 \wedge \cdots \wedge T_n)|$ is a linear combination of $\sin^i\sqrt{c}\,t\cos^{n-i-1}\sqrt{c}\,t$ or of $\sinh^i\sqrt{-c}\,t$ $\cosh^{n-i-1}\sqrt{-c}\,t$ or of $1, \cdots, x^{n-1}$. In any of these cases there is a (unique) differential equation of order n with constant coeficients satisfied by any such combination. The same equation would hold for h_t applied to the unit $(n-1)$-vector at any y in M, and therefore also for A_M, since integration over M will commute with differentiation by t.

Added in proof. More general results have been announced in the author's paper, *Riemannian manifolds of finite order*, Bull. Amer. Math. Soc. **78** (1972) 200–201.

References

[1] R. A.. Holzsager & H. Wu. *A characterization of two-dimensional Riemannian manifolds of constant curvature,* Michigan Math. J. **17** (1970) 297–299.

[2] H. Wu. *A characteristic property of the Euclidean plane,* Michigan Math. J. **16** (1969) 141–148.

AMERICAN UNIVERSITY